Lev B.

Do you know of great mathematicians that were *not* Platonist? You said Bourbaki group were anti-Platonist, but I wonder if a) how "great" Bourbaki's contribution to math really was and b) if there were mathematicians at the very top of the mathematical Parnassus who were anti-Platonist.

Dear Lev

Well, Bourbaki's *initial* objective just was to establish solid foundations for analysis. A bit later, the group realized that from their perspective, such an objective needed deeper efforts to unify mathematics on the basis of set theory. Even if the planned work *Elements de mathématique* never was achieved, the Bourbaki group made relevant contributions concerning the actually written unification of mathematics. Namely analysis was founded on algebra and topology, both being directly derived from set theory.
I think that the Bourbaki group in substance tried an approach going absolutely in the sense of Hilbert's ideas: *re*(!!)constructing *given* mathematics in order to consolidate it by explicitly human-made rigorous frameworks. Now, it is difficult to understand why the Bourbakists presenting themselves as authentic followers of Hilbert, *de facto* defend caricatural "formalism" Hilbert never would have cautioned. Perhaps the answer is that during the period going back to the twenties and continuing until the eve of the third millennium, "metaphysics" was almost a swearword. Under these conditions, in the eyes of many people, neo-positivistly inspired caricatural formalism was the sole possible way to escape "metaphysics." Dieudonné perhaps was not sufficiently explicit about Hilbert's idea to reconstruct mathematics *as if* it was "formalist", while others probably did not ask too many "metaphysical" questions.

Carl K.

My question [is] how does the platonic math reality interface with the human mind? The implication of the question is that there must be a joining and reduction of subject object distinction. Leads to phenomenology /existential analysis as well as cognitive science etc. Thanks for the presentation.

Dear Carl,

Let us approach the issue of subject/object distinction in the opposite sense. Phenomenology, or existential analysis and so on are subjective philosophy. Authors belonging to phenomenology generally do not agree among them. Concerning existential analysis, Jean-Paul Sartre, according to my purely subjective idea, sometimes is contradicting himself, but anyway, neither he nor his main reference Martin Heidegger escape endless discussions. Having a degree in cognitive science, I am not so sure that cognitive science – globally speaking – is science. From my standpoint, cognitive science *when touching final issues like "What is objective and what is subjective?"* encounters the same problems as German idealism – say Kant, Fichte, Schelling, Hegel – going from difficulty to difficulty and subsequently from discussions to discussions.
Now, there is a big contrast between the forgoing and mathematical edifices. Despite Gödel's theorems and so on, mathematics holds astonishingly well. Whereas Sartre's disciples continuing to claim that existence precedes any essence always are contradicted by Sartre's adversaries asserting that any essence precedes existence, no serious mathematician – regardless of her/his *philosophical* school – would say something like "In 'my opinion', such or such theorem is not a theorem." And when a lack is discovered within a mathematical proof, making from a theorem a non-theorem, it is in turn not a question of "opinion." Perhaps we have to take into account the position of some extremists which, belonging to (the) "constructivist" school(s), do not recognize certain theorems considered as "non-constructive." Well, but that which constructivism(s) and other schools commonly recognize as theorems represents a very substantial subset of "classical mathematics", while no philosophical field – say phenomenology, existential analysis, hermeneutics, esthetics and so on – can be sure that a substantial subset of its assertions is beyond discussion.
Controversies related to mathematics systematically concerns the foundations of mathematics, say in this context that which makes from mathematics a special case.
So, *since* we consider mathematical as something intellectual, the special status of mathematics implies a special conception of the subject/object distinction implying in turn (i) that objectively existing objects are not necessarily *material* objects and (ii) that interfaces between cognition and objectively existing *material* objects must be different from interfaces between cognition and objectively existing *immaterial* objects.
Now, of course, there *are* different cognitive modalities corresponding to the cognition dedicated to material objects *or* the cognition dedicated to something outside empirical data *stricto sensu*. Of course, the cognition dedicated to mathematical entities and their relations is not the same as material perception. By contrast, *nothing* justifies the current conception that mathematical entities needing specific forms of cognition would necessarily be "created" by the latter.

Matthew A.

In the talk it was emphasized that mathematical entities exist independent of the human mind. Someone else then asked something along the lines of "how do we 'download' this mathematical knowledge into our brains".
I want to consider how Socrates may have answered this based on his concept of 'knowledge as remembering' discussed in Plato's Phaedo. Socrates argues that (paraphrasing to keep this question reasonably short…) when we see two similar things (sticks, rocks…) they remind us of equality, even though no matter how similar two material things are they will ultimately fall short of being truly equal. But if the only things we observe are material things, how can we be 'reminded' of equality by them (that is how do we already have knowledge of equality)? Socrates answer is that at one point we knew true equality before we were born but upon doing so, forgot it, and so invokes the notion of a soul that existed and had wisdom before we were born.
It is not a stretch to include mathematics in this. For example every attempt to construct a perfect physical right triangle will fail yet we are reminded of right triangles in countless geometry/physics problems. This is how it seems Socrates might have answered how we 'download' mathematics (we were all great mathematicians before we were born!). At the same time this argument probably will make many modern scientific 'skeptics' cringe. So is there any modern day alternative to how we come to have knowledge of mathematics?

Dear Matthew,

First, trying to answer Carl's question, I also tried to show that "downloading modalities" dedicated to material things being necessarily different from "downloading modalities" dedicated to immaterial things are not a problem. We can add that this difference has nothing to do with the possibility, nor the alleged impossibility of objectively existing immaterial objects. Let us do *as if* – to paraphrase Hilbert – mathematics was a kind of chess organized by arbitrary but internally consistent rules and so on. Even under those conditions, the cognitive faculties/capacities a good teacher of mathematics would try to mobilize among her/his students in order to help the latter to understand mathematics are certainly different from cognitive faculties/capacities to be mobilized with regard to geology, and this *independently* from the fact that a geology teacher *de facto* is not obliged to prove the "objective existence" of the objects she/he is supposed to explain, whereas the ontological status of mathematical entities remains controversial.
On the other hand, I entirely agree that Plato's reminiscence theory exposed in the Phaedo dialogue, despite its undeniable beauty is not the best way to convince a modern scientist about Platonism. Even general philosophers not especially interested in philosophy of science often consider dialogues like the Phaedon as a mythologico-philosophical one, rather than philosophy *tout court*. Anyway, saying that our soul was a great mathematician before being incarnated in matter belongs neither to science, nor to philosophy of science. Add that many voices are claiming that Plato probably would not recognize himself in all that which is said in the name of mathematical/scientific Platonism, yes, but people working seriously on what is called "mathematical/scientific Platonism" generally know it. There are several approaches used in order to defend mathematical/scientific Platonism without reminiscence theories presupposing that our souls were great mathematicians before being incarnated in our bodies.
In my skype presentation I tried my own approach: (i) *If* a given theory belongs to metaphysics, *then* all its possible negations also belong to metaphysics. (ii) Mathematical/scientific Platonism obviously belongs to metaphysics. So it is the same for its negations. In other words, negations of mathematical/scientific Platonism are not more – nor less – "scientific" than mathematical/scientific Platonism itself. (iii) Subsequently, the Platonism v/s anti-Platonism debate is metaphysics v/s metaphysics. (iv) However, even if mathematical/scientific Platonism as well as anti-Platonism is metaphysical, we can compare both options according to criteria developed by modern philosophy of science which have nothing to do with metaphysics. To make short, we can compare the number of hypotheses needed by both approaches, the simplicity/complexity of these hypotheses, their scientific foundations and so on. (v) To presuppose the objective existence of a consistent immaterial reality is a heavily metaphysical hypothesis. But the forgoing is not less heavily metaphysical than the hypothesis presupposing the objective existence of the material world. (see below) We are perhaps – if we want – better familiarized with the material world than with the immaterial mathematical world, knowing nevertheless that "familiarity" is rarely a determining reference, neither in science, nor in philosophy. (vi) If we refuse to accept the hypothesis of an objective existence of a immaterial mathematical reality, this does not mean that we escape the not less metaphysical hypothesis presupposing the objective existence of the material world. But, above all, this refusal would be payed by plethora of really farfetched additional hypotheses.
To illustrate the forgoing, let us retake your argument concerning a perfect physical right triangle. Well, we say that we never will manage to draw a perfect physical right triangle, but how could we know by exclusively empirical means whether our triangle is perfect or not? Comparing a lot of actually drawn allegedly right triangles, we probably realize that all among them are at least slightly different from any other, but this is not incompatible with the eventuality that by chance one among the allegedly right triangles is really perfect. And even if this *would* be the case, we cannot determine which one is the good. Of course, there is the "idea" of a perfect right triangle. But, independently of the question whether this idea is a Platonist one, or merely a idea in

the current sense of the term, issued from our brains, how to compare a physical, actually drawn triangle to an idea?
Well, apparently – but appearances can be deceptive – there is a solution. We could draw "a very great number" of "approximately" perfect right triangles and then "determine the mean." The latter would be the reference allowing us to evaluate and to correct the others. But in fact, "a very great number" of "approximately" perfect right triangles, in our context has no value. To obtain a *mathematically* perfect right triangle, we have to operate over an *infinity* of "approximately" perfect right triangles. And saying "infinity", we necessarily mean *actual infinity* being so scandalous in the eyes of anti-Platonists. Nevertheless, for to be able to accept just *potential infinity*, we need a *given* perfect right triangle to which our empirical ones could converge, but since we try to *obtain* this perfect right triangle, we cannot consider it as "given." Perhaps certain people now reply that finally a "very good approximation" of a perfect right triangle also would do the job. But remember that the perfect right triangle is the support of the good old Pythagorean theorem which is in turn a simple special case of a quadratic form, and without rigorous – not "very well approached", but rigorous – quadratic forms, a vast part of mathematics would not be mathematics, knowing that mathematics intrinsically comprising non-mathematics would be nonsense. And so on; there is an infinity of additional problems, beginning with the law of great numbers which, concerning very different things like roulette or the rentability of insurances, subsequently must be immaterial, unless we accept genuine circularity.
Now let us come back to the above-mentioned geologist who *de facto* can work without complicating her/his life with ontology. And yet, the objective existence of the very tangible reality concerning a geologist raises a lot of philosophical questions: How can I be sure that the allegedly existing world is not an illusion of my poor mind? How can you be sure that the letters a, d, y, … belonging to this text have the same meaning for you and for me? Since great authors like Kant or Hegel tried to find a solution for such issues, the latter probably cannot merely be reduced to a "caricature of philosophy." However, on the one hand, a geologist who actually want to do her/his job has no other choice than to accept the hypothesis claiming the objective existence of the material world. On the other hand, the acceptance of this hypothesis does not affect the scientific quality of the work done by a geologist.
I think that for mathematicians or mathematical physicists, the situation is analogous. The acceptance of the objective existence of the mathematical entities and their relations never had hampered a mathematician to do her/his work. On the contrary, if for example, G. Cantor had not courageously assumed his Platonism, despite it being incomprehensible in the eyes of people like Kronecker, he would not have reached the results he reached. So, why a mathematician should refuse the objective existence of the mathematical entities and their relations and pay this refusal by a plethora of farfetched, in fact indefensible hypotheses?

Al B.

Let me follow-up on Matthew A comments above. I am assuming the implication here is that mathematical entities exist in some kind of undefined "ether" (if you will). If mathematical entities exist independent of the human mind in some type of ether, wouldn't this existence require some sort of specification and set of requirements for this media? Substrates for the concept of DNA exist in the neurons of our brains, in the scientific literature, and many other places. If you subtracted our neurons, scientific literature, & other locations, where specifically do the mathematical entities exist? (May be it's in some undefined space between the multiple universes.
If we based our conclusions on the starting point of mathematical entities exist independent of the human mind, how can this starting point be falsified and a better replacement be found? Also, how are the beauties of mathematical entities connected to this undefined ether and to the entities themselves? And what exactly is mathematical beauty?

Dear Al,

If we reduce any existence to *material* existence, yes, *then* we also have to admit that "mathematical entities exist in some kind of undefined 'ether'", presupposing that the "ether" in question is "material" and so not entirely undefined. Now, I agree that an immaterial existence can seem mysterious, but material existence in turn is far from being self-evident. As a Christian believer, I believe that our material world was created by God, but scientifically speaking and avoiding any mixture of faith and science, I am not able to prove that the existence of our material world is an ontological necessity. Of course, there are people who do not share religious faith. So there are – often interesting – discussions between believers and non-believers, but the sole fact that of such discussions are undertaken proves that the existence of the material world is not self-evident at all. As I recalled it in my skype presentation, Leibniz asks why our world does exist instead of non-existing? Could you prove "scientifically" that Leibniz's question is nonsense? So, accepting (i) the existence of the material world despite all the mysteries this existence carries, and (ii) this other mystery that the material world is astonishingly intelligible to us, there is no reason (i*) to exclude *a priori* any immaterial existence because of its own mysteries and (ii*) to refuse that such an immaterial existence is in turn intelligible, but by other ways than the material one.
Concerning DNA, yes, our brains are equipped to conceptualize it, to make scientific literature about it, and so

on. But DNA as such already was present at the level of primitive life, long before the occurrence of humans able to conceptualize it. So, the existence of DNA does not presuppose the existence of human brains with their extraordinary conceptualization abilities, nor is "supported" by human brains. It is the same for mathematics. Only human brains are able to conceptualize mathematical entities and their relations, to make scientific literature about and so on. But this point does not imply that mathematics cannot exist outside human brains, nor outside scientific literature. But *if* we assert that mathematics is created, or constructed by human brains, *then* we also have to assume that the DNA of, say, dinosaurs was created or constructed by human brains long before the appearing of humans.
Concerning scientific falsification, no scientist can falsify the ontological, so metaphysical hypothesis that our *material* world exists, instead of it being an illusion, and so on. A scientist only can falsify a *hypothesis about a phenomenon* belonging to the material world supposed existing. A mathematician can falsify an alleged theorem by discovering a non-consistency within its proof. But as well as no scientist is able to falsify the existence of the material world, no mathematician could do it with the Platonist world.

Mathematical beauty is approached in my answer to Alexey's question.

Kevin S.

Perhaps it would be good for me to clarify the question I had asked earlier. In my question, I supposed the existence of a universe where Euclidean geometry is not useful. The speaker pointed out that Euclidian geometry is a subset of Riemann geometries, which he claims have some existence outside of physical reality.
I would say that there is an even larger set of geometries that obey the basic rules of arithmetic (i.e. 1 + 1 = 2). However, one could imagine a mathematical system in which addition was not a useful operation. Perhaps another operation, f(x,y) = x + y + g(x,y) is more relevant to the laws of nature in another universe. We could let g(2,2) = 1 and g(x,y) ~= 0 for values far away from 2,2.
So, in some sense, we could have a universe in which 2 + 2 = 5. Perhaps there could be other values as well as (2,2) where g(x,y) deviated significantly from zero. Or we could imagine a universe in which counting numbers are not even a relevant quantity. Nimbers (or grundy numbers) are an alternative to counting numbers, and perhaps there could be a universe in which they are the fundamental building blocks of the mathematics that is relevant to physics.
The point is that, if mathematical axioms are themselves neither true nor false, we can still imagine physical systems in which they're not useful. A hard-nosed philosopher could insist that 2 + 2 =4 is still true in this other universe, their laws of physics just don't use addition the way ours do. But this just seems to be relabeling the same thing in different language. If you're going to talk about abstract math as it interfaces with actual science, like physics or biology, you can't ignore what is applicable to that actual science, and is therefore "true" or not, in a more pragmatic sense.

Dear Kevin,

I think, to begin, we have to retake the context of my argument touching Euclidean geometry as a special case of Riemannian pangeometry. In my presentation, I merely used the fact that Euclidean geometry is a special case of Riemannian pangeometry as an example of the *embedding issue* which, from my standpoint, represents a particularly intuitive argument in favor of Platonism.
Simplifying a bit, we can say that there are mathematical edifices E_i being embedded in an edifice E_{i+1}, the latter eventually being embedded in E_{i+1} and so on. On the other hand, *there are* edifices E_i, E_{i+1}, $E_i \subset E_{i+1}$, where we are authorized to say: *If* E_i holds, *then* E_{i+1} also holds. *If* E_{i+1} does not hold, then E_i in turn does not hold. Gödel's second theorem prevents us from establishing this *if–then* relation for any E_i, E_{i+1}, $E_i \subset E_{i+1}$. This *if–then* relation must be proved for any individual case. But there are cases of E_i, E_{i+1}, $E_i \subset E_{i+1}$ where it had been proved that the *if–then* relation in question holds. We will come back to this point.
Now, let us suppose that there are E_i, E_{i+1}, $E_i \subset E_{i+1}$, satisfying this *if–then* relation, while from a strictly *historical* perspective, E_i is known by humans, and E_{i+1} not. On this basis, my argument in favor of Platonism is the following: *If* we say that E_i is "constructed" by humans so that E_i is consistent, *then* the "constructors" also must "construct" E_{i+1}, $E_i \subset E_{i+1}$, since the consistency of E_{i+1} is necessary for the consistency of E_i. But how could these "constructors" "construct" E_{i+1} without knowing that they are constructing it? I think that without very farfetched presuppositions, there is no solution. By contrast, saying that E_i is *discovered*, while E_{i+1}, whose consistency is necessary for consistency of E_i, was discovered later seems defensible with more credible presuppositions.
In my presentation I evoked Saccheri, whose misadventure at the beginning of the 18^{th} century illustrates the forgoing. Saccheri tried to prove Euclid's parallel postulate *per reductio ad absurdum*. Replacing the postulate by its two formally possible negations, he hoped to show that this operation would create absolute non-consistency within geometry. But he found no non-consistency within geometry. Without knowing it, Saccheri made non-Euclidean geometry which, from this perspective, existed already before being *knowingly discovered*. But, above all, that which *objectively* prevented Saccheri from managing his proof, *objectively* existed before being *knowingly discovered*.

Before coming to your own arguments *I entirely share*, we just have to recall that the forgoing statement concerning the embedding-issue is simplified and so reducing. It goes together with Saccheri's misadventure, but the general case needs an extension.
Instead of saying "E_i, E_{i+1}, $E_i \subset E_{i+1}$, so that *if* E_i holds, *then* E_{i+1} also holds and *if* E_{i+1} does not hold, then E_i in turn does not hold.", let us consider mathematical edifices E_r, E_u which *as such* do not express *common* intrinsic properties. Note $P(E_r)$, $P(E_u)$, … the (sets of) properties characterizing respectively E_r, E_u, … Now, on the one hand, if any $P(E_r)$ had nothing to do with any $P(E_r)$, $r \neq u$, mathematics would be a senseless juxtaposition of all kinds of things. On the other hand, all attempts to "found mathematics on *one* ultimate E°" – for Bertrand Russel, E° was logic, for Bourbaki set theory – have failed. And yet, there is a unity of mathematics we can characterize as follows. To begin, say that two mathematical edifices E_r, E_u, so that $P(E_r) \cap P(E_u) = \emptyset$, express unity if there is an edifice E_{ru} so that $P(E_r) \cap P(E_{ru}) \neq \emptyset$ and $P(E_u) \cap P(E_{ru}) \neq \emptyset$. To go further, admit that there is an edifice E_z and an edifice E_{uz} so that $P(E_u) \cap P(E_{uz}) \neq \emptyset$ and $P(E_z) \cap P(E_{uz}) \neq \emptyset$, whereas – roughly speaking – $E_{ru} \neq E_{uz}$. Under these conditions, (E_r, E_u) and (E_u, E_z) express unity, but can we assert the same for (E_r, E_z)? From the perspective of classical first order transitivity, the answer is no, but defining a n-order transitivity implying a hierarchy of edifices E_r, E_{ru}, $E_{ru.....}$, we can evoke n-order unity.
On this basis, we can conceive mathematics as a set of edifices where each edifice is related to each other by a n-order transitive relation. This leads to minimalist conception of unity of mathematics, but this conception is payed by a high Platonist price: edifices constructed by humans – or other "constructors" living in other universes – while supposed to be consistent would imply that these "constructors" simultaneously "construct" a quasi-infinite or perhaps strictly infinite number of hierarchized edifices without knowing what they construct.
Now let us come to your arguments.
If, concerning $f(x,y) = x + y + g(x,y)$, "+" is to be understood as a current addition, whereas $x,y \in \mathbb{R}$, then it is easy to prove that if ($\mathbb{R}$,+) holds, then ($\mathbb{R}$,+,+ g(x,y)) also holds, and that if ($\mathbb{R}$,+,+ g(x,y)) does not hold, ($\mathbb{R}$,+) in turn does not hold. Indeed, ($\mathbb{R}$,+) as a special case of ($\mathbb{R}$,+,+ g(x,y)) – with $\forall(x,y)$, $g(x,y) = 0$ – is embedded in ($\mathbb{R}$,+,+ g(x,y)). Now, let us retake your universe where ($\mathbb{R}$,+,+ g(x,y)) goes together with physical experience. Under the pressure of experience, as Riemann would say, the inhabitants of this universe initially are convinced that ($\mathbb{R}$,+,+ g(x,y)) is the unique possible algebra. But perhaps there is a mathematician worthy of the name, knowing that experience just can be a first trigger of mathematical work which, thereafter, has to forget experience. This mathematician wants to *prove mathematically* that ($\mathbb{R}$,+,+ g(x,y)) is the unique possible algebra. But as well as Saccheri, she/he would encounter great difficulties leading to the *discovery* of an infinity of existing algebras, and the eventual fact that *our* familiar ($\mathbb{R}$,+) is physically speaking useless in the considered universe does not change anything about the fact that ($\mathbb{R}$,+) was there before being *discovered*, and this is the sole thing to be taken into account within the Platonism v/s anti-Platonism-debate.
Concerning your formulation "let g(2,2) = 1 and g(x,y) ~= 0 for values far away from 2,2", its rigorous approach would need an exact definition of "g(x,y) ~= 0" and "values far away from 2,2." Probably, such an edifice represents difficulties with regard to direct embedding. In this case, to belong to mathematics, this edifice must express n-order unity with all other edifices. A mathematician inhabiting the considered universe examining whether this extension of ($\mathbb{R}$,+,+ g(x,y)) is the unique possible algebra would have much more work than her/his colleague working on ($\mathbb{R}$,+,+ g(x,y)) as such, but the Platonist/anti-Platonist challenge would remain the same.
Note that ($\mathbb{R}$,+,+ g(x,y)) is not very far from Riemannian pangeometry embedding Euclidean geometry.
Consider a quadratic form $g_{ij}a^ia^j$. Obviously, $g_{ij}a^ia^j$ is Euclidean if $g_{ij} = 1$ for $i = j$ and $g_{ij} = 0$ for $i \neq j$. Certain voices referring to the Minkowski framework of SR defined over $\mathbb{C}$ qualify as "pseudo-Euclidean" a quadratic form $g_{ij}a^ia^j$ with i,j = 1, …, 4, where indifferently $g_{11} = g_{22} = g_{33} = -1$, with $g_{44} = 1$, or $g_{44} = -1$, while $g_{11} = g_{22} = = g_{33} = 1$. However, from this standpoint, any configuration of $g_{ij}a^ia^j$ where for i = j, a given g_{ij} can equal 1 or –1, should be qualified as "pseudo-Euclidean." In all other cases, $g_{ij}a^ia^j$ is strictly non-Euclidean. Once again, *here* the *argument in favor of Platonism* is the following. (i) Since the *infinite* set of all $g_{ij}a^ia^j$ forms the support of a transformation group, we can say and have to say: *If* Euclidean geometry holds, *then* any case of non-Euclidean or pseudo-Euclidean geometry also holds. *If* we find at least one case of non-Euclidean or pseudo-Euclidean geometry which does not hold, *then* no geometry hold. (The exact proof of the forgoing is due to Felix Klein also reaching the same result by his famous model of non-Euclidean geometries.) (ii) So, *if* Euclidean geometry is " consistently constructed by humans" , for example "starting from experience to be generalized" or so, *then* its "constructor", unconsciously, perhaps despite her/himself, had *simultaneously* "constructed" an *infinity* of geometries being not accessible to "experience." And *if* the "constructor had not "constructed" an *infinity* of geometries, *then* Euclidean geometry is not consistent. (iii) If (without italics) we find it too complicated to evoke a "constructor" of consistent Euclidean geometry "constructing" simultaneously an infinity of geometries without knowing it, then (without italics) a conception of mathematics considered as objectively existing, despite its own difficulties, is perhaps more convincing.

As the forgoing finally is longer than anticipated, let us very quickly consider Grundy numbers. From my standpoint, nimbers are a good example showing that the *discovery* of initially unknown mathematical edifices implied by known ones not necessarily passes by the to reference to other physical universes.
Without entering into details, say that classical game theory is very reducing. There are games necessitating extensions of classical game theory. Now, for these extensions not to be foreign corps within mathematics, there are the above-mentioned two solutions: either, classical game theory is directly embedded in the considered non-classical one, or there is an edifice E_{ru} so that $P(E_r) \cap P(E_{ru}) \neq \varnothing$ and $P(E_u) \cap P(E_{ru}) \neq \varnothing$, where E_r, E_u are for classical game theory and the considered non-classical one. So, in both cases, we meet once again the embedding-issue encountered by mathematics considered as "constructed" – indifferently in the formalist or constructivist sense – without needing other physical universes than ours taken in sense being not very near of physics.

Vitaly P.

Hi Peter (and Alexey). I am not deeply knowledgeable in your particular topic, but despite I am fearing to look a Lack of all trades, let me ask a question that would help me clarify what you actually have in mind in your paper. You "collide" Platonism and anti-Platonism, not so widespread opposition nowadays. Are not you, in fact, discussing the positivist (or logical empiricist, if you wish) and post-positivist positions, because the former implies anti-Platonism, and the latter – Platonism, but also many other important things in between (for example the Kantian transcendental apperception to "bind" Platonic forms to our consciousness to form theories)? In other words, is not your Platonism-anti-Platonism opposition in fact empiricism-post-positivism one (widely discussed in past decades, with plenty of arguments and ended up by the victory of the latter)? Just to understand.

Dear Vitaly,

You are absolutely right saying that both positivism v/s anti- or post-positivism and Platonism v/s anti-Platonism debates do overlap. But this overlapping does not prevent that both debates are, with regard to their respective challenges, essentially different.
Simplifying a bit, the purpose of positivism is to show that metaphysics is senseless. So, the debate about positivism is a debate concerning the philosophical legitimacy – or non-legitimacy – of metaphysics.
The debate around mathematical Platonism concerns the issue whether mathematical edifices exist objectively or not.
Of course, mathematical Platonism as such belongs to metaphysics. So positivism refuses mathematical Platonism, but not exclusively mathematical Platonism. Positivism, as you mention it very rightly, is against each form of metaphysics comprising among others Kant's transcendental philosophy. Carnap's *Überwindung der Metaphysik durch logische Analyse der Sprache* especially attacks Heidegger's existential analysis, Reichenbach tries to demolish neo-Kantism from which he is issued himself.
Now, the debate between Platonism and anti-Platonism is *not* a debate between metaphysics and anti-metaphysics, nor between metaphysics and non-metaphysics. The debate between Platonism and anti-Platonism is a confrontation – *within* metaphysics – of different metaphysical conceptions.
Of course, positivism cannot be consistent. Or, more precisely, to remain consistent, positivism must leave aside many essential things. As a first approximation, positivism says that mathematics is "just a language" bringing as such nothing but tautologies. Only experience could determine whether mathematical languages are adequate with regard to the physical world. However, *no* language could be adequate with regard to a physical world functioning "anyhow". This point implies ontological issues pushing positivism to contradict itself. On the other hand, positivism seems to forget that its choice to consider metaphysics as nonsense makes that its efforts to "prove" the adequacy of this choice in turn lead to nonsense.
Well, here is not the place to "refute positivism". I entirely agree that the days of glory of positivism are over, and that there is no more need to discuss positivism. But I think that precisely the decline of positivism allows to approach the Platonism-anti-Platonism debate under more favorable conditions, *knowing that this debate continues to be an essential philosophical issue*.
Since, consecutively to the decline of positivism, "metaphysics" is no longer a pejorative word, people more and more accept that the negation of a metaphysical theory is in turn a metaphysical theory. So, instead of "being for Platonism" or "against Platonism", we can quietly compare Platonism to its nor more, nor less metaphysical negations. These negations comprise mathematical formalism, constructivism, naturalism, but also Kantism and so on, without forgetting that certain representatives of "cognitive sciences" merely retake the main theses of German idealism, just replacing "spirit" and "reason" by "brain" and "cognitive faculties" or something like that, inheriting in this way all the difficulties German idealism had encountered.
Concerning my own presentation, its purpose was just to show that according to non-metaphysical criteria currently carried by philosophy of science, Platonism, despite its own difficulties, is much easier to defend than anti-Platonism, needing in all cases substantially farfetched presuppositions.

Alexey B.

Dear Peter, in your consideration I do not see any place given to the mathematical beauty. Does it mean that you see it unrelated to the problem of mathematical ontology? Would you agree that in the Platonic World various mathematical structures/ideas exist independently of their beauty, that their beauty is something totally human? Do you think that the 'mathematical democracy' of Tegmark, which Lev and I refuted for the physical multiverse, can be accepted for the Platonic world, instead of the 'mathematical aristocracy' of the beautiful systems?

Dear Alexey,

I am sure that we entirely agree concerning mathematical beauty related to mathematical ontology.
But concerning my presentation, the big problem was to make as short as possible, in order to not exceed 45 minutes. The logical configuration of the presentation having been the comparison of Platonism and its current negations with regard to criteria like the respective number of required hypotheses, the complexity of these hypotheses, the scientific quality of their derivation, … I was led to point out very *compact* while *striking* problems like circularity or really farfetched hypotheses as we find them within anti-Platonism. Any reference to mathematical beauty would have required a lot of secondary developments. Probably, it would be better to treat mathematical beauty as a subject *per se*, or as the main argument in favor of a given thesis, whereas in my presentation this argument, by necessity, would have been reduced to an underdeveloped argument among others.
Well, mathematical beauty, as well as esthetic beauty in the common sense, cannot be defined, but we can conceive it through data being inseparable from a Platonist conception of mathematics.
Mathematical beauty is – among others – symmetry, knowing that without symmetry, physics would not be physics.
Mathematical beauty is – among others – the fact that many theorems can be proved in several independent ways; this seems incompatible with issues constructed by humans.
Mathematical beauty is – among others – the simplicity and shortness of its proofs, and so its intelligibility.
This latter point, perhaps allows to make a significant link between mathematical Platonism and Platonism *tout court*. For Plato himself, beauty and truth are nearly exact synonyms, but on the other hand, beauty according to Plato is not that what is commonly understood by this concept, namely in the context of fine arts since Plato says that the representation by material means of beauty as pure idea necessarily is degrading. So, from the standpoint of Platonism *tout court*, the concept of beauty *focuses* on truth while carrying an esthetic connotation recalling moral beauty. Now, concerning any moral dimension, *mathematical Platonism* has nothing to do with it. But recognizing within the concept of mathematical beauty a kind of esthetic dimension going further than esthetics in its current sense, the link, within mathematical Platonism, between mathematical beauty and mathematical truth becomes rather explicit: without mathematical beauty, mathematical proofs would be unintelligible, whereas mathematical beauty makes that mathematical proofs are more, infinitely more than that what police or justice as human institutions mean by a proof.
Concerning Tegmark's mathematical democracy, I think that from a Platonist standpoint, each mathematical edifice belongs to mathematics, and that within mathematics there is only consistency, neither aristocracy, nor democracy, and this because:
Mathematical beauty is – among others – the fact that Mathematical beauty is – among others – the fact that we find links between in principle essentially different edifices, and this within contexts where we do not expect it.
By contrast, mathematics as such forms an aristocracy within knowledge.

Dear Alexey, I have not forgotten your other question. They are here below.

A question asked several times from the audience.

This question is about whether a conclusion about existence of that timeless Mind is a necessary consequence of existence of the Platonic World? Does existence of the Mathematical Platonic World require existence of Mind to whom this world belongs. If not, where does it exist?

I think, here first we have to elucidate the issue weather the existence of any world necessarily *belongs* to any form of mind. To begin, let us write "mind" with a small "m" denoting our human minds. And, forgetting for the moment the Platonist world, let us just consider our material world. Does the latter "belong" to our minds? Or does it exist independently of our minds?
Both questions are the subject of an endless debate which, absolutely speaking, cannot be decided by our limited minds. There is a lot of attempts to answer both – complementary – issues, but no one is entirely satisfying. Berkeley, for example, says that things exist by the sole fact of being perceived. Aware of potential inconsistencies within his philosophical system – a house with closed window shutters whose inhabitants have gone out would not exist in its interior while existing at its exterior when passers-by perceive it, and so on – Berkeley makes intervene God "who is perceiving everything.". Kant, a Christian believer nevertheless thinking

that philosophy as such should not confuse religious faith and pure reason qualifies this kind of solution as God of the gaps argumentation. Being myself a Christian believer, I share, *in this context*, entirely Kant's position. But the solution proposed by Kant is in turn problematic. Kant says that our world does objectively exist; however, we only know this world re-configured by the faculties of our mind, and not as it is *per se*. There is something true: We cannot assert that the wall facing us "is" white. All we can know is that our perception is made in a manner that we perceive the wall as white. However, Kant's approach is problematic since he asserts the objective existence of a world *per se*, while asserting simultaneously that we cannot know anything about this world. So Fichte says that the world emanates from our *ego*, without explaining how our personal perceptions of a world emanating from a multitude of *egos* can converge sufficiently, so that we can at least communicate. On the other hand, Fichte's system dangerously approaches solipsism. Hegel tries to find a solution which, remaining unintelligible for a lot of students, is highly speculative. And so on.

So, for scientists *as such*, the best approach can be summarized as followed: (i) We note that the world exists, without asking whether this world "belongs to our mind(s)" or not. (ii) We realize that this position – non-naive realism assumed as the least worst solution despite its philosophical difficulties – does not hamper scientific work, whereas mixing science *and* questions like "Does the material world belong to our mind(s)?" actually *would do it*. (iii) We nevertheless remain aware that our material world which exists also could not exist, and that ultimately, the existence of our material world is a mystery.

Now, if we accept the existence of our material world despite its mysteries, then there is no reason to exclude *a priori* the existence of an immaterial word expressing existence modalities essentially different from the material one.

Personally, I think that the conception of mathematics as an objectively existing immaterial world is more consistent than all the known alternatives whose consistency require really farfetched presuppositions. So we can accept the existence of the Platonist mathematical world without asking whether it belongs to *any* mind, since we are not doing anything different concerning our material world. On the other hand, if, overly simplifying, things of our *material* world exist in space and time, this does not imply the legitimacy of questions like "*Where* does the Platonist mathematical exist?" And finally, such an over simplification is not a good idea. It is *within* our physical universe that things exist *within* in something, ultimately *within* space-time. But the sole question within what the whole universe exists has no sense, and even multiverse speculations would not change anything about this point. So, why must just a Platonist mathematical world exist *within* something, or *somewhere* or belong to something? Anyway, concerning specifically the Platonism v/s anti-Platonism debate, such questions are extrapolations from the *material* world to the immaterial one, and this *regardless of whether the latter actually does exist*.

From my standpoint, within philosophy of science *stricto sensu*, there is no other answer to the question "Does existence of the Mathematical Platonic World require existence of Mind to whom this world belongs. If not, where does it exist?"

The other issue "whether a conclusion about existence of that timeless Mind is a necessary consequence of existence of the Platonic World?" only can be treated if we accept to leave science *stricto sensu*, turning to *theology* interested in its potential relations *with* philosophy of science. Personally, as a Christian believer, I am interested in this area, precising nevertheless that, whereas a believer can feel her/his faith strengthen and even confirmed by scientific knowledge and above all by the interpretation of scientific knowledge which seems to her/him more adequate than others, everybody is free to share religious faith or not. In other words, while it would not be a good idea for theology presupposing faith to remain outside scientific knowledge as a working process, scientists and/or philosophers of science being not interested in theology can remain consistent without becoming interested in it.

This being said, if by "timeless Mind" we understand God, we have to precise that God is timeless in this sense that He transcends his own Creation comprising time. Saint Augustine had said that God creating the world also had created time; so question like "What was *before* the creation of our world?" are senseless. But on the other hand, God is in relation with His Creation and even can change these relations. So God, without any non-consistency, is timeless and non-timeless.

Now there are theologians – Richard Swinburne, William Lane Craig – according to them the latter point *collides* with Platonism conceiving an immaterial, eternal and unchanging world. If God is the Creator of all things – material and immaterial ones – then the Platonist world, against its definition, cannot be eternal, nor unchanging, since the passage from non-being to being is change. And if the definition of the Platonist world is adequate, then the Platonist world is not created and in this case God could not be the Creator of all things.

Personally, I try to develop a different approach. According to Plato himself, the mathematical world is just a basic aspect of the eternal and immaterial world where, according to the author, we can find truth. But, the exploration of the mathematical world, precisely because of its relatively basic aspect, is the first step of the exploration of the entire Platonist world, a first step helping humans to understand that beyond the "intuitive" material world, there can be an immaterial one, and that anti-materialism, or non-materialism is more consistent

than primary materialism.
So, why should a believer not suppose (i) that that what we call the Platonist mathematical world regroups timeless/eternal aspects of God, and also a very small part of what God has in his Mind, (ii) and that God, by His free decision, makes accessible to sufficiently motivated people some aspects of His immaterial and eternal existence and His infinitely consistent Mind.
From this perspective, the existence of the mathematical Platonist world, *logically/epistemologically* speaking, does not require the existence of a Mind, but from a metaphysical/theological perspective, the conception of a mathematical Platonist world *emanating* from God can consolidate the consistence of a theological edifice open to epistemology and philosophy of science, but ultimately based on faith.
Nevertheless, since a mathematical Platonist world emanating from God is not a logical necessity for the the consistency of Platonism, the latter can be shared by people which do not share faith. And proposals like "For Platonism to be consistent, it must be something like religion" advanced for example by Hartry Field, are not relevant.

Lev B.

Dear Peter, thank you very much for the talk. My question is about the value of your suggested metaphysics. Suppose that we extend naturalism to some primordial substance with two different aspects: the ideal and material, which are somehow interrelated with each other. Would that be a satisfactory resolution of the difficulties posed by naturalism as presented in your talk? If so, what do we really gain by this extension? After all, the formulas will remain the same either way, as are the methods and experiments established by the real practice of science and scientists, won't they?

Dear Lev,

Penelope Maddy is considered as the most representative author belonging to naturalism; on the other hand, it is also Penelope Maddy who is saying that "naturalism" means a lot of things, sometimes contradictory. So, in my relatively short presentation – 45' is not much time for a complicated subject like Platonism – I was obliged to advance the most consensual conception of naturalism: a philosophical approach "operating according to the model of natural sciences." Now, of course, a very naive conception of the acquisition of mathematical knowledge says that the observation of natural phenomena allows the discovery of laws, that we can "idealize" these laws, that these "idealized" laws are a first step to mathematics, that these first step laws can be "generalized", and so on. The forgoing is perhaps a caricature, but P. Maddy says nothing other, and personally, I do not see another way to approach the essence of mathematics while "operating according to the model of natural sciences." Leaving beside the fact that pre-mathematical knowledge – yes, everybody knows that the Egyptians, Babylonians … already had discovered the Pythagorean theorem by empirical means – is not mathematics, since mathematics as an activity begins with the decisions of the Greeks to forget experience, I tried to show in my presentation that the above-mentioned naturalist view of "mathematics" is undermined by circularity: To be "idealizable" and "generalizable", natural phenomena cannot "behave" anyhow. If we want to obtain mathematical truth by the observation of natural phenomena, we previously have to suppose that the latter behave in a way to converge to what we pretend to obtain by idealization.
Now, an extension from Platonism to a kind of neo-Platonism presupposing a primordial substance comprising two aspects, one immaterial and one material, would not change anything concerning this circularity. On the other hand, only a small part of presently known mathematics is interpreted by physical laws.
So, the simplest way, from my standpoint, to conceive (i) the foundations of mathematics and (ii) the relations between mathematics and physics remains the following: First, realizing that every kind of anti-Platonism required really farfetched presuppositions which in principle do not go with the spirit of scientific work. And then, simply saying that Platonism as a non-farfetched approach of the foundations of mathematics implies mathematical truth preceding *ontologically* physical laws. People who do not agree are challenged to try *any* computer simulation of a physical process without considering previously given mathematical truth.
I am sure that you share these proposals as a co-author of GPU refuting chaosogenesis.

Alexey B.

Cher Peter,

Merci beaucoup pour votre discours et la citation.

Lev comment below is correct. My question was:

Do you know any great mathematician who clearly denied Platonism even at its minimal variant, like natural numbers only? I understand, that there were some unclear situations, but the question was about distinctive indubitable denial. Did Dieudonne reject even the minimal Platonism 'on Sundays'?

Dear Alexey,

“Do you know any great mathematician who clearly denied Platonism even at its minimal variant, like natural numbers only? I understand, that there were some unclear situations, but the question was about distinctive indubitable denial. Did Dieudonne reject even the minimal Platonism 'on Sundays'?”
first, I think that constructivism – perhaps with some rare exceptions; Kronecker's position on natural numbers “given by God” is not clear; his German formulation “*der liebe Gott*” maybe was ironical – cannot be qualified as minimal Platonism. Among constructivist theses, you can find a lot of variations: “Mathematics is constructed on the basis of some given 'intuitions'” – but it is not precised where the “given intuitions” are coming from, nor by what these intuitions are “given”, or “Mathematics is constructed on the basis of some given data and or functions anchored in the human brain”, or “A mathematical entities “exists” if there is a “given” algorithm leading to it”, without explaining where this algorithm is coming from, and so on. All that has nothing to do with Platonism, even not with a minimal.
Now, there were great mathematicians sharing constructivism and so radical anti-Platonism: Brouwer, Heyting, Gentzen, Kolmogorov, Apéry, Reeb and others.
Concerning Dieudonné, as I tried to explain it in my answer to Lev's first question, in fact a reformulation of your question. Probably, Dieudonné never was an anti-Platonist. I think, he understood the Bourbaki approach as an *as-if reconstruction* of mathematics, which is perfectly compatible with Platonism.

Alexey B

Dear Peter,

Thank you so much for your detailed answers to the questions of the audience. They are helpful for me, and I think the other people will find that as well.

If you do not mind, I'd like to ask a bit more with my last question on this row, about existence of great mathematicians who clearly denied Platonism. Your statement about anti-Platonism of constructivism does not sound convincing to me, because, as far as I understand, constructivism is a purely methodological creed, related to what can be accepted as a proof and what cannot. It has nothing to do with ontology, to my mind. Maybe I am wrong on that, so please correct me in that case.

Do you know any direct citation of a high rank mathematician with a clear denial of mathematical Platonism on one or another ground?

Dear Alexey

Let us *provisionally* say that constructivism *could* be a purely methodological *approach*, more precisely an attempt to reconstruct existing mathematics. Such an approach *could* be operated without any ontological concern. In other words, an approach focusing exclusively on the consolidation of mathematical frameworks would not be obliged to ask in which way mathematical entities and their relations exist.
Indeed, a minority of constructivists consider constructivism as a reconstruction of mathematics on – according to them – more solid grounds than those chosen by Hilbert. I think that this conception of constructivism is more or less identical with the contemporary tendency saying that constructivism and axiomatic approaches are not incompatible.
But here, heavy difficulties arise, making that formulations like “Constructivism *could* be a purely methodological *approach.*” in fact can not be more than a kind of *provisional* introduction to the problems *preventing* constructivism from being a purely methodological *approach*. And we will see that these problems imply the soundness of your choice to evoke a methodological *creed*, and not a methodological approach.
An axiomatic conception of mathematics – regardless of its Platonist, anti-Platonist or other orientations – sees the “existence” of a mathematical entity and its relations as given if this entity and its relations are consistent with regard to the admitted axiomatic system. On the one hand, there *is* a potential discrepancy between constructivism and *strictly* axiomatic conceptions of mathematical *existence*. Even if modern constructivist tendencies *try* to deny these discrepancy, the founding fathers of constructivism were not wrong while asserting the contrary. On the other hand, the axiomatic conception of mathematical *existence* already represents an *ontological* choice which, far from being minimalist, has been and continues to be the subject of deepened investigations. So, the defense of the opposite – constructivist – conception of an axiomatic conception of mathematical existence implies in turn deepened investigations on the *ontological* level.
Briefly speaking, beyond purely methodological motivations, the constructivist choice *is* an ontological position. Note that these general considerations *as such* have not automatically Platonist, nor anti-Platonist consequences.

However, passing from general considerations to precise issues, the choice between constructivism or non-constructivism immediately implies an ontological choice between anti-Platonism and Platonism. Indeed, the Hilbertian axiomatic conception of mathematical *existence* necessarily the existence of the actual infinity. Now, from a Platonist standpoint, an actual infinity is not a problem. But from a constructivist standpoint, an actual infinity, by definition, is unthinkable. More technically, an axiomatic conception of mathematical *existence* going sufficiently far – but finally not *so* far –necessarily finishes to be confronted to the acceptance or non-acceptance of the ZFC axiomatic set theory and so of two intimately related things: the continuum hypothesis and the axiom of choice.
Zermelo's axiomatization (Z) confers consistency to set theory as such, avoiding namely in a credible way set theoretic paradoxes. (This point will be retaken.) But if we want to consider set theory as a framework supposed to found mathematics in general, then Z is not sufficient and must be completed.
This extension of Z, due to Fraenkel (and others like Skolem) is called Zermelo-Fraenkel set theory (ZF)
Now, there are issues, beginning by the continuum hypothesis (see below), whose approach necessitates ZF completed by the axiom of choice. The ZF system completed by the axiom of choice is called ZFC.
However the axiom of choice is highly controversial. From an explicitly Platonist standpoint, the axiom of choice does not pose any problem. But from a non-Platonist standpoint, the axiom of choice obviously is a farfetched ad-hoc-hypothesis.
Gödel has showed that the continuum hypothesis can not be refuted within ZFC. In the sixties, it was shown by Cohen that the continuum hypothesis can not be deduced within ZFC.
In other words, the continuum hypothesis is non-decidable within ZFC; so its status within ZFC is equivalent to status of Euclid's parallel postulate within geometry. This point is among the greatest results of 20th century mathematics. But this result presupposes the *ontological* choice to accept the actual infinity.
Being "not-constructive", the actual infinity is refused by *all* constructivist tendencies. This choice – in turn ontological – amputates analysis from its own foundations and assimilates " constructivist analysis" to a naive reduction of continuity to density.

You asked me an anti-Platonist quote emanating from a high ranked mathematician. Well, in order to introduce a highly significant anti-Platonist quote, let us return to Z, successfully elaborated by the Platonician Zermelo in order to eliminate the paradoxes striking naive set theory. Now, Brouwer, considered as the founding father of constructivism, had in turn elaborated his own approach – intuitionism – in order to eliminate the set theoric paradoxes. According to Brouwer, these paradoxes result from the non-constructive aspect of Cantor's approach accepting the actual infinity. So, the – in principle – methodological choice to remain in the area of constructable, leads Brouwer to the following assertion: "*Thus, for instance, it has nowhere been proved that a finite number, subjected to a provably consistent system of contradictions, must always exist.*" (It is in Brouwers PhD thesis)
Far from being just a methodological choice, intuitionism, to remain consistent, has to advance ontological decrees difficult to defend.

I also have a more explicit quote of Brouwer, but it is second hand quote, due to Hesseling's translation from the Dutch, a language I a understand more or less, but not more than that.
"*The only thing that is real to me is my own self at the moment, surrounded by a wealth of images in which the self believes and which make the self live. The question whether these images are "factual" is devoid of meaning: for my self, only the images exist and are, as such, real. A second reality, independent of my self and corresponding to these images, is out of question.*"

Heyting, Brouwer's most influent disciple says: "*If 'to exist' does not mean 'to be constructed', it must have some metaphysical meaning. It cannot be the task of mathematics to investigate this meaning or to decide whether it is tenable or not. We have no objection against a mathematician privately admitting any metaphysical theory he likes, but Brouwer's program entails that we study mathematics as something simpler, more immediate than metaphysics.*"

Of course, the classical texts of constructivism does not explicitly evoke "mathematical Platonism" : this expression was introduced later, by Hilbert's collaborator Bernays. But I think the forgoing quotes are significant. Heyting claims that he is against any form of "metaphysics", comprising Platonism, neglecting that any approach of "existence" is ontology, so metaphysics. Brouwer openly decrees his own ontological conception which does not go very far: It is the usual schematized idealism common to people who – contrarily to great representatives of idealism like Kant or Hegel, aware of the potential difficulties of their philosophical systems – did no deepen the context of their own approaches. For Brouwer, it would be hard to reject convincingly the blame of solipsism implying mathematics changing from one individual to another.